Poznámky - geometrické reprezentace grafů I

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Definition 1 (Intersection graph)**.** Let M be a family of sets. Then the intersection graph of M is $IG(M) := (M, \{\{m, m'\} : m \cap m' \neq \emptyset, m, m' \in M\})$.
The class of all intersection graphs of a family of families is $\mathcal{IG}(\mathcal{M}) := \{IG(M) : m \in \mathcal{M}\}$.

Definition 2 (Classes INT, CA, CIRCLE, PER, FUN, OUTER-STRING, SEG, CONV, P$_{hom}$, STRING, PC, IFA)**.**

- INT= $\mathcal{IG}$(intervals on a line)
- CA= $\mathcal{IG}$(arcs on a circle)
- CIRCLE= $\mathcal{IG}$(chords of a circle)
- PER= $\mathcal{IG}$(segments between two parallel lines)
- FUN= $\mathcal{IG}$(graphs of continuous functions)
- OUTER-STRING= $\mathcal{IG}$(curves in a plane with their one endpoint on a common line)
- SEG= $\mathcal{IG}$(segments in the plane)
- CONV= $\mathcal{IG}$(convex sets in the plane)
- P$_{hom}$ $= \mathcal{IG}$(homothetic copies of a polygon P)
- STRING== $\mathcal{IG}$(curves in the plane)
- PC== $\mathcal{IG}$(polygons inscribed on the circle)
- IFA= $\mathcal{IG}$(inverval filaments of a line)

Theorem 1. PC$\subset$IFA

Proof. Take a k-polygon and convert it into a filament with k half-circles, where the touching points are the images of the polygon's vertices. ⊞

Definition 3 (Chordal graph)**.** G is chordal, if every cycle of length at least four has a chord.

Definition 4 (Simplicial vertex)**.** $u \in V(G)$ is simplicial, if $G[N(u)]$ is a clique.

Lemma 1 (Vertex cuts in chordal graphs)**.** Every inclusion-wise maximal vertex cut in a chordal graph induces a clique.

Proof. By contradiction: take such cut A so that $G-A$ is disconnected and let there exist $x \neq y \in A : xy \notin E$. Then there are components $C_1, \ldots, C_n$ of $G - A$.

There must exist $u_1 \in C_1 : xu_1 \in E, u_2 \in C_2 : xu_2 \in E, v_1 \in C_1 : yv_1 \in E, v_2 \in C_2 : yv_2 \in E$. By connectedness of the components, there exist paths $u_1 \to v_1, u_2 \to v_2$ - take the shortest ones for P_1, P_2. We get a cycle in G by $x, u_1, P_1, v_1, y, v_2, P_2^{-1}, u_2, x$ - take the shortest cycle, which has no chords (no edge between components, in components by the shortest path) and it is an induced cycle of length at least 4. ⊞

Lemma 2 (Chordal graphs and simplicial vertices)**.** If G is chordal, then it is either isomorphic to a clique, or it contains two nonadjacent simplicial vertices.

Proof. By induction on n: for $n = 1, 2$ or G complete - simple. Induction step: G is not complete, hence there exists a vertex cut. Let A be a minimal vertex cut and $C_1, \ldots, C_k$ the components. Take $G_1 = G[A \cup C_1], G_2 = [A \cup C_2 \cup \ldots \cup C_k]$. Both are chordal on less vertices, and hence they have simplicial vertices.

Both G_1 and G_2 have two simplicial vertices, one of which is $u \in C_1$ and G_2 has a simplicial vertex $v \notin A$. Therefore $uv \notin E(G)$, and $N_G(u) = N_{G_1}(u), N_G(v) = N_{G_2}(v)$, and we have two nonadjacent simplicial vertices in G ⊞

Definition 5 (Perfect elimination scheme)**.** A graph G has a perfect elimination scheme if it has a linear order of its vertices $v_1, \ldots, v_n$ such that $\forall i \in [n] : v_i$ is simplicial in $G[\{v_1, \ldots, v_i\}]$.

Theorem 2 (Chordal graphs and PES)**.** Every chordal graph has a perfect elimination scheme.

Proof. By induction on n. For $n = 1$, this is trivial (only one order exists).
For $n > 1$, G has a simplicial vertex, which we take as the last one, and get the rest of PES by the induction hypothesis. ⊞

Definition 6 (Perfect graph)**.** A graph is perfect, if for every induced subgraph, its clique number is the same as its chromatic number.

Theorem 3 (Chordal graphs are perfect)**.** Chordal graphs are perfect.

Proof. Greedily coloring (First Fit) by PES using the smallest color - if we use k-th color, there must be a k-clique. ⊞

Definition 7 (Clique-tree decomposition)**.** Given G, let $Q_1, \ldots, Q_l$ be all maximal cliques (with respect to inclusion). A clique-tree decomposition is a tree $T = (\{Q_i : i \in [l]\}, E)$, such that $\forall v \in V(G) : T[\{Q_i : v \in Q_i\}]$ is connected.

Theorem 4 (Alternative characteristics of chordal graphs)**.** The following are equivalent:

1. G is chordal
2. G has a PES
3. G allows a clique-tree decomposition
4. G is an intersection graph of subtrees of some tree

Proof. $1 \Rightarrow 2$ by a previous theorem.

$3 \Rightarrow 4$: Take T a clique-tree decomposition. For a vertex u there is a subtree $T_u := T[\{Q_i : u \in Q_i\}]$. Apparently $T_u \cap T_v \Leftrightarrow uv \in E(G)$. $uv \in E \Leftrightarrow \exists Q_i : u, v \in Q_i \Leftrightarrow Q_i \in T_u \cap T_v \Leftrightarrow T_u \cap T_v \neq \emptyset$

$2 \Rightarrow 3$: Given a PES, we construct a clique-tree decomposition by induction. Take a $v_1, \ldots, v_{n-1}$ and take a T_{n-1} its clique-tree decomposition of $G \setminus \{v_n\}$.

Now take $Q = N(v_n)$. If Q is a maximal clique in G_{n-1}, then just add v_n there. Otherwise, let $Q' \supset Q$ be a maximal clique, and attach $Q \cup \{v_n\}$ to Q'.

$4 \Rightarrow 1$: Given a tree T, we will show that there may not exist an induced cycle of length at least 4. Suppose there exists a cycle $c_1, \ldots, c_k$. Then the subtrees T_1, T_2 must have a nonempty intersection, the same for T_2, T_3 and T_k, T_1. At the same time, there must be an empty intersection between T_1, T_3 and T_2, T_k. Hence, the trees T_2, T_k live in different connected components of $G - T_1$. Therefore, T_3 up to T_{k-1} must live in the same connected components. However, then T_k has to live in $G - \bigcup_{i<k} T_i$ and intersect T_1 at the same time, which is not possible, as the graph is disconnected by then. ⊞

Definition 8 (Lexicographic BFS ordering)**.** A lexicographic BFS ordering of a graph $G$is a left-to-right ordering of its vertices $v_1, \ldots, v_n$ such that for any $v_i, v_j : i < j$, if there is a vertex $w \in \{v_1, \ldots, v_{i-1}\}$ adjacent to exactly one of v_i, v_j, then the leftmost such w is adjacent to v_i.

Algorithm 1 (LexBFS)**.** Q - a queue of lists of vertices (ideally a linked list)
$Q :=$ a single list containing w Repeat until $Q \neq \emptyset$:

1. choose a vertex w from the first list in Q and remove it from the list.
2. for any list L in Q containing a neighbour of w : split L into two lists L^+, L^-, where $L^+ = L \cap N(w)$ and $L^- = L \setminus L^+$.
3. insert L^+, L^- into Q to the position of L (unless one of them is empty)
4. output w as the next vertex in LexBFS ordering

Lemma 3 (LexBFS property)**.** Suppose $v_1, \ldots, v_n$ is a LexBFS ordering of G. Suppose $v_a, v_b, v_c \in V$ such that $a < b < c, v_a v_c \in E, v_a v_b \notin E$. Then, there exists $d < a$ such that $v_d v_b \in E, v_d v_c \notin E$.

Proof. Follows from the definition, as otherwise v_c would have a higher priority over v_b. ⊞

Proposition 1 (LexBFS ordering of a chordal graph)**.** If G is chordal, then any LexBFS ordering of G is a PES.

Proof. Let G be chordal, let $v_1, \ldots, v_n$ be its LexBFS ordering and suppose its not a PES.

Choose $a < b < c$ such that $v_a v_b \notin E, v_b v_c, v_a v_c \in E$ so that a is as small as possible. From the previous lemma, there exists $b_2 < a$ such that $v_{b_2} v_b \in E, v_{b_2} v_c \notin E$ - take the left-most possible. Then $v_{b_2} v_a \notin E$, or a C_4 would exist. Then we have the triple b_2, a, b so from the lemma, we get a_2 such that $v_{a_2} v_a \in E, v_{a_2} v_{b_2} \notin E$.

This may be repeated ad infinitum, which is a contradiction with finiteness of the graph. ⊞

Lemma 4 (Not-PES condition)**.** If $v_1, \ldots, v_n$ is not a PES, then there exists a triple $v_a, v_b, v_c : v_a v_b \notin E, v_a v_c \in E, v_b v_c \in E$ with v_b being the right-most neighbour of v_c to the left of v_c.

Proof. Choose v_a, v_b, v_c such that $v_a v_c \in E, v_a v_b \notin E$ so that $b - c$ is smallest possible. Claim: v_b must be the rightmost left neighbour of v_c.

If not, let $v_{b'}$ be the rightmost left neighbour. Then, there exists a better structure for a, b, c given by a, b, b'. ⊞

Algorithm 2 (Linear-time chordality checking)**.** Each vertex $v \in V$ has a list TODO(v), initially they are all empty For $k = n$ to 1:

1. Let v_j be the rightmost left neighbour of v_k, add all the other left neighbours of v_k to TODO(v_j)
2. Make a mark on all left neighbours of v_k
3. Go through TODO(v_k) and check if they are marked (if not, fail)
4. Remove the marks.

return true

Lemma 5 (Algo time complexity for linear-time chordality checking)**.** The algorithm takes time $\mathcal{O}(m + n)$.

Proof. For a given k, 1 takes $\mathcal{O}(\deg(v_k))$.
2 takes $\mathcal{O}(\deg(v_k))$
3 takes $\mathcal{O}(|\text{TODO}(v_k)|)$
4 takes $\mathcal{O}(\deg(v_k))$
When summed together, the lengths of todo lists are exactly $|E|$, as every edge means a single addition into a todo list. ⊞

Definition 9 (Interval, cyclic interval)**.** For a sequence $s_1, \ldots, s_t$, a set $S \subset \{s_1, \ldots s_t\}$ is an interval, if $S = \{s_k : k \in \{i, i+1, \ldots, j-1, j\}\}$ for some i, j.
A set S is a cyclic interval, if it is an interval or $\{s_1 \ldots, s_n\} \setminus S$ is an interval.

Theorem 5 (INT iff path-clique)**.** G is an interval graph if and only if the maximal cliques of G can be ordered into a sequence $Q_1, \ldots, Q_t$ such that for every vertex $x \in V$, the cliques containing x form an interval in (Q_i).

Proof. From the sequence, we get the interval representation by taking intervals corresponding to the indices of the cliques.

If G is an interval graph, we take every maximal clique in the representation and take a "cut" through the intervals. All cliques containing x form an interval by the fact that they all lie on an interval representing x. ⊞

Proposition 2 (Number of maximal cliques in a chordal graph)**.** Let $G = (V, E)$ be a chordal graph. Then, G has at most n maximal cliques and their list can be completed in time $\mathcal{O}(m + n)$.

Proof. Let $v_1, \dots, v_n$ be a PES. Define $N_L[x] = \{x\} \cup \{\text{left neighbours of} x\}$. By the definition of PES, $N_L[x]$ is a clique. Moreover, any maximal clique is in the form of $N_L[x]$ for some vertex x (there exists a right-most vertex in the PES). Hence there are at most n maximal cliques.

In time $\mathcal{O}(m+n)$, it is possible to compute all the sets $N_L[x]$ and the remove from the list all such sets which are not maximal. ⊞

Definition 10 (Testing of Intervals, Testing of Cyclic Intervals)**.** Testing of Intervals: IN: number t, sets $S_1, \dots, S_n \subseteq [t]$
Question: is there a permutation $\pi \in \mathbb{S}_t : S_i$ are intervals for all i on the permutation?

Testing of Intervals: IN: number t, sets $S_1, \dots, S_n \subseteq [t]$
Question: is there a permutation $\pi \in \mathbb{S}_t : S_i$ are cyclic intervals for all i on the permutation?

Observation (TCI and TI)**.** TI can be reduced to TCI by taking the input $t, S_1, \dots, S_n$ for TI and getting the answer for TCI with input $t+1, S_1, \dots, S_n$.

Definition 11 (Cyclic shift, cyclic equivalence, cyclic permutation)**.** A cyclic shift is an operation transforming $\pi_1, \dots, \pi_t$ into $\pi_t, \pi_1, \dots, \pi_{t-1}$.

Two permutations are cyclically equivalent, if one can be transformed to the other by a sequence of cyclic shifts.

A cyclic permutation is an equivalence class of cyclic equivalences.

Definition 12 (PQ-tree)**.** A PQ-tree T is a tree, whose leaves are numbered $1, 2 \dots, t$ for some t and whose internal nodes have one of two types: P or Q, and each internal node has a prescribed cyclic permutation of its incident edges.

Then, we define $\pi(T)$ as the cyclic permutation of $[t]$ induced by the leaves of T.

A PQ-tree T' is said to be equivalent to the PQ-tree T, if it can be obtained from T by a sequence of operation of two types:

1. change the cyclic permutation of edges incident to a P-node to any other cyclic permutation
2. replace the cyclic permutation ef edges incident to a Q-node with its mirror image

Then we define $\mathcal{P}(T) := \{\pi(T') : T' \text{ is equivalent to } T\}$.

Theorem 6 (Testing of cyclic intervals)**.** Let $(t, S_1, \dots, S_n)$ be an instance of TCI. If the instance has a solution, then there exists a PQ-tree T whose $\mathbb{P}(T)$ is equal to the set of solutions of the instance. Moreover, the tree T can be found in time polynomial in $t + \sum |S_i|$.

Proof. By induction on n. For $n = 0$, the tree has a single P node. For $n = 1$, the tree consists of two P nodes, one of which has all the leaves which lie in S_1 and the other has the remaining leaves. Let $n \geq 2$ and suppose we have a PQ-tree T^- representing the solutions of the TCI instance $(t, S_1, \dots, S_{n-1})$. Let $S = S_n$.

We will say a subtree of T^- is full, if all of its leaves belong to S, empty if none of its leaves belong to S and mixed otherwise.

An edge e of T^- is mixed if both components of $T^- - e$ are mixed. Note that if e, e' are mixed, then all edges on the path from e to e' are mixed as well.

If there is a vertex x of T' adjacent to more than two mixed edges, then no cyclic permutation on $\mathbb{P}(T)$ has S as a cyclic interval and therefore the TCI instance has no solution. Otherwise suppose the mixed edges form a path $P = e_1, \dots, e_k$. Reorder the edges adjacent to P in order to obtain T' equivalent to T^- in which all the empty subtrees are above the path P and all the full ones are below. If this is impossible, then there is no solution.

Then for every P-node on P, we change the P-node into a Q-node with four neighbors - two from the path, and two more P connected to the that the “above” subtrees and the “below” subtrees.

After that, there are only Q-nodes on P, so we contract the path into a single Q-node. The resulting tree represents all the solutions of the instance. (In case $|P| = 0$, we just convert and contract if necessary.) ⊞

Definition 13 (Comparability graphs)**.** G is a comparability graph if and only if there exists a partial order $\leq$ of V, such that if $xy \in E(G)$, then $x \leq y \vee y \leq x$.

Proposition 3 (COMP are perfect)**.** Comparability graphs are perfect.

Proof. For a poset, we have the Dilworth (and Mirsky) theorem: minimum number of chains covering the poset is the maximum size of an independent set, and the minimum number of antichains covering the poset is the maximum size of a clique. As an antichain is an independent set, we have that the maximum size of the clique is the minimum number of independent sets to cover the poset, hence we get perfectness. ⊞

Theorem 7 (FUN=CO-COMP)**.** FUN=CO-COMP

Proof. FUN⊆CO-COMP: There is an induced poset on $\forall x \in [0,1] : f(x) > g(x)$. If this inequality does not hold, then there exists $x_0 : f(x_0) = g(x_0)$, and therefore there must be an edge and vice versa.

CO-COMP⊆FUN: Given a partial order P on V, we construct functions f_v on $[0,1]$. Take k linear orders L_i such that $P = \cap L_i$. Then we define $f_v(x)$ such that we take k vertical lines and on the i-th line, $f_v(\frac{i-1}{k-1}) = |\{u \in V : u < v\}|$ and define the curves to be linear. This yields a FUN graph. ⊞

Theorem 8 (PER=COMP∩CO-COMP)**.** PER=COMP∩CO-COMP

Proof. PER⊆FUN=CO-COMP.

Next, we show PER=CO-PER. Take a permutation graph G of π. Then, take a graph of the reverse permutation $\sigma(i) = \pi(n-i+1)$. If $i > j$, then $\pi(i) > \pi(j) \Rightarrow \sigma(i) < \sigma(j)$ and $\pi(i) < \pi(j) \Rightarrow \sigma(i) > \sigma(j)$, hence we get the complement of G.

Therefore CO-PER⊂CO-FUN=COMP.

Now, COMP∩CO-COMP⊆PER. Let G ∈COMP∩CO-COMP – we have a transitive orientations F_1 of E and F_2 of $\binom{V}{2} \setminus E$. $F_1 \cup F_2$ is a linear order: antisymmetry and antireflexivity easy, transitivity: cases with uv in F_1, vw in F_2.

Also $F_1^{-1} \cup F_2$ is a linear order, as F_1^{-1} is a transitive orientation. Take two vertical lines $y = 0, y = 1$, and linear functions $f_v(0) = |\{u : (u,v) \in F_1 \cup F_2\}|$, $f_v(1) = |\{u : (u,v) \in F_1^{-1} \cup F_2\}|$ ⊞

Theorem 9 (INT=CHOR∩CO-COMP)**.** INT=CHOR∩CO-COMP

Proof. INT⊂CHOR, as every subpath on a path is a subtree of a tree. INT⊂CO-COMP, as we take the poset on vertices, where $u < v$ if and only if $I(u)$ is to the left of $I(v)$.

The other direction follows from the following theorem. ⊞

Proposition 4 (Characterisation of interval graphs)**.** The following are equivalent:

1. G ∈INT
2. $G \in IG$(subpaths of paths)
3. G has a clique-path decomposition
4. G ∈CHOR∩CO-COMP

Proof. $1 \Rightarrow 4$: easily

$2 \Rightarrow 1$: easily

$3 \Rightarrow 2$: easily

$4 \Rightarrow 3$: Let $Q_1, \ldots, Q_n$ be all maximal cliques with respect to inclusion. The complement of G is also transitively oriented by relation P. Define $Q_i < Q_j \Leftrightarrow \exists u \in Q_i, \exists v \in Q_j : (u,v) \in P$. — TODO

⊞

Proposition 5 (CHOR⊆ PC)**.** CHOR⊆ PC

Proof. Given a chordal G, there exists a tree T and their subtrees induces by the vertices. Draw the tree and have all sub-trees have the leaves in the leaves of T. Then, draw a circle around T and build a k-agon M_u with vertices in $T_u \cap C$.

If $T_u \cap T_v \neq \emptyset$, then there exists a vertex which is in both, and therefore the intersection of the two polygons is nonempty.

On the other hand, if $T_u \cap T_v = \emptyset$, then there exists an edge, which separates the two subtrees, hence there exists a curve in C which separates the two polygons. ⊞

Definition 14 ($\mathcal{G}$-mixed)**.** Let $\mathcal{G}$ be a class of graphs closed on induced subgraphs. Then $G \in \mathcal{G}-$mixed if and only if we can partition $E(G) = E_1 \sqcup E_2$ and transitively orient E_2 such that $(V(G), E_1) \in \mathcal{G}$ and $\forall x, y, z : (x, y) \in \overrightarrow{E_2} \wedge yz \in E_1 \Rightarrow xy \in E_1$.

Theorem 10. If Weighted Clique is in P for $\mathcal{G}$, then Clique is in P for $\mathcal{G}$-mixed.

Theorem 11 (CO-IFA=(CO-INT)-mixed)**.** CO-IFA=(CO-INT)-mixed

Proof. CO-IFA⊆(CO-INT)-mixed: $G \in$IFA $\to \overline{G} \in$CO-IFA. There will be two types of edges: one above the other, and two next to each other – the first type will be E_2, the second E_1. E_2 gives a natural orientation and E_1 is a complement of an interval graph.

(CO-INT)-mixed⊆CO-IFA: Get $G = (V, E_1 \sqcup E_2) : (V, E_1) \in$CO-INT - we get $(V, \overline{E_1}) \in$ INT, so we get intervals I_v for $v \in V$. If $I_x \subseteq I_y$, then $xy \notin E_1$. If I_x is to the left of I_y, then $xy \in E_1$. If they intersect and the leftmost point of I_x is more left than the leftmost point of I_y and the other cases have not occurred, then $xy \notin E_1$.

However, it may happen that $xy \notin E_2$ and they have nonempty intersection and $I_x \not\subseteq I_y \not\subseteq I_x$. But then, it's always possible to change the intervals to $xy \in E_2 \Rightarrow I_x \subseteq I_y$. Another issue: $I_x \subseteq I_y$ but $xy \notin E_2$ and hence $xy \in E$. This can be fixed by making the filament higher. ⊞

Remark (Clique, MIS, WIS, WClique, Coloring on INT)**.** Clique, MIS, WIS, WClique and Coloring are in P for INT

Remark (Clique on COMP)**.** Clique is the largest chain in the poset – partition the poset - take minimal elements and cut them off.

Remark (Independent Set on COMP)**.** By Dilworth theorem, the size of the largest antichain in the poset is the smallest k such that the poset can be partitioned in k chains.

G can be partitioned into k vertex-disjoint paths if and only if $\overrightarrow{G}$ has a set of $n-k$ edges such that every vertex has at most one outgoing and at most one ingoing edge in S, which happens if and only if $\overrightarrow{G}$ has a matching of size $n - k$. Then $\overrightarrow{G}$ is a bipartite graph on vertices v_{in}, v_{out}, where $x < y \Rightarrow (x_{out}, y_{in}) \in E$.

By König's theorem, the number of edges in maximum matching equals the number of vertices in minimal vertex cover and given a vertex cover of size $n - k$, then $A := \{x \in P$:neither x_{in}, x_{out} are in $C\}$ is an antichain: if there was a pair of comparable vertices, then one of them woiuld have to be in the cover.

Remark (Clique on IFA)**.** Largest clique in IFA is a largest IS in CO-COMP.

Definition 15 (Cops-and-robber game)**.** Given a connected undirected graph G, $c(G)$ is the minimal number of cops necessary to catch a robber on a graph in finite time.

For a class $\mathcal{A}$, $c(\mathcal{A}) = \max_{G \in \mathcal{A}} c(G)$.

Proposition 6 (Some cop number values)**.** $c(\text{PATHS}) = 1$
$c(\text{CYCLES}) = 2$
$c(\text{TREES}) = 1$
$c(\text{INT}) = 1$
$c(\text{CHOR}) = 1$

Theorem 12 (Cop number of IFA)**.** $c(\text{IFA}) = 2$.

Proof. Two policemen: one is a hunter, the other is a guard. The guard stands over all the filaments in which the robber is. The hunter walks on the upper envelope of the other filaments and walks to the right until he either catches the robber or the robber's filament is below the hunter - then the two switch their roles. ⊞

Definition 16 (Constrained OUTER-STRING)**.** A Constrained OUTER-STRING graph is an outerstring graph with additional ordering of the endpoints of vertices on the common line.

Theorem 13 (Complement of C_k is not constrained OUTER-STRING)**.** The complement of a k-cycle (with $k \geq 4$) and the ordering $1 \ldots, n$ along the cycle is not a constrained OUTER-STRING graph.

Proof. $\overline{C_4} = C_4$ - simple. For $C_n, n \geq 5$: Let it be representable, and therefore it has a finite number of intersections. Take the representation with the least amount of intersections. Take a curve number 1, which we shorten by their last intersection point, from which we lose an edge in the graph. At least one of the resulting is a complement of a smaller cycle, which is a contradiction with the induction hypothesis. ⊞

Theorem 14 (Hasse and OUTER-STRING)**.** For G which is K_3-free: G is a Hasse diagram if and only if $\overline{G}$ is outer-string.

Proof. If $\overline{G} \in$ OUTER-STRING - then we take the order based on the points touching the baseline, which is an acyclic orientation. Assume there exists a cycle in the complement which cannot exist in Hasse diagrams - there exists an induced cycle, however its complement may not be represented as an outerstring.

The other inclusion: G is Hasse - there exists an acyclic orientation, we take the topological order and using induction on the highest point, we get an outer-string representation. ⊞

Remark (Independent Set on IFA)**.** Define $F_i : I_i \to [0, +\infty)$ and so on. Assume if $F_i < F_j(I_i \subseteq I_j, \forall x \in I_i : F_i(x) < F_j(x))$, then $i < j$.

Define $IS(j)$:=the size of the largest independent set under F_j.

For $i = 1$ to $n+1$ (assume all of the filaments are under F_{n+1}): take G induced by intervals for F_i under F_i and find weighted independent set on the interval graph with weights $w_j = IS(j) + 1$ for I_j under the interval graph and this induces the result.

Return $IS(n+1) = \alpha(G)$.

Theorem 15 (Circle graphs are χ-bounded)**.** Circle graphs are χ-bounded.

Proof. Take two special configurations: K_j and $Q_{k,l}$, where the second is an indepedent set of size l under the complete graph on k vertices. Define $m_{j,k,l}$:= the larges possible chromatic number of a graph with a representation without both K_j and $Q_{k,l}$. Want: $m_{j,k,l} < \infty$.

Induction on $k : m_{j,0,l}$ is bounded – let G have a circle representation without both K_j and $Q_{k,l}$. Greedily find a sequence of chords $P_1, \dots, P_i$, where P_{a+1} is completely to the right of P_a and whose right endpoints are as much to the left as possible. Surely $i < l$. Take G_a := graph induced by the chords whose left endpoint is to the left of the right endpoint of P_a, but not to the left of the right endpoint of P_{a-1}.

Note that the union of the vertices of G_i is $V(G)$.

Every G_a is a co-comparability graph, and therefore is perfect (COMP are perfect and perfect graphs are closed on complement), and therefore we may color it using j colors as it does not contain K_j. Hence we may color G using lj colors and $m_{j,0,l} \leq lj$.

For $k \geq 1$: we will show $m_{j,k,l} \leq m_{j,k-1,2l+1}$. Take a G with representation without forbidden $K_j, Q_{k,l}$ and let x be a vertex represented by a chord, whose left endpoint is at most to the left. Define G_a as a subgraph of G induced by vertices in distance a from x (WLOG G is connected). There exists an edge between G_a and G_b iff $|a - b| \leq 1$. Every vertex in G_a for $a \geq 1$ has a neighbour in G_{a-1}.

We will show that for every a, G_a does not contain K_j nor $Q_{k-1,2l+1}$. By contradiction: Let G_a contain $Q_{k-1,2l+1}$, and let y be its middle vertex (of the $2l+1$ stable set). Let $P = x, x_1, x_2, \dots, x_{a-1}, y$ be the shortest path from x to y in G. Surely $x_i \in G_i$. x_i must lead from the midst of y out of the $Q_{k-1,2l+1}$ configuration, which yields $Q_{k,l}$.

Therefore $\forall a : G_a$ can be colored by $m_{j,k-1,2l+1}$ colors, and we may use it to color $G_0, \cup G_2 \cup \dots$ by $m_{j,k-1,2l+1}$ and $G_1 \cup G_3 \cup \dots$ by another set of $m_{j,k-1,2l+1}$ colors, therefore we may color G by $2m_{j,k-1,2l+1}$ colors. ⊞

Theorem 16 (SEG is not χ-bounded)**.** SEG graphs are not χ-bounded. (Precisely: there exists a triangle-free SEG graph with an arbitrarily large chromatic number.)

Proof. Define S_i as a graph, for which there exists a probe, that contains i colours for every valid coloring of S_i.

Inductively: S_1: a single line segment. $S_{i-1} \to S_i$: take the root of every probe, add a copy C of S_i and for each of the copies, for every probe, create a new segment cutting across the probe and split the probe in two, one containing the original probe's vertices and the other containing the new vertex (only). ⊞

Definition 17 (AT-graph, strongly/weakly realizable)**.** An AT-graph is a tuple $((V,E),R)$ for $R \subset \binom{E}{2}$.
An AT-graph is strongly realizable if there exists a planar drawing D of (V,E) such that two edges e, f intersect if and only if $\{e,f\} \in R$.
An AT-graph is weakly realizable if there exists a planar drawing D of (V,E) such that if two edges e, f intersect, then $\{e,f\} \in R$.

Lemma 6 (AT-graph and the amount of intersections)**.** Let $((V,E),R)$ be an AT-graph and let D be its weak realisation with the minimum number of intersections. Then if $D(e)$ intersects k edgs, then there are at most $2^k - 1$ intersections on $D(e)$.

Proof. Let there be an edge e with at least 2^k intersection points which intersects edges $e_1, \dots, e_k$. We direct the edge and for every interval between two intersection points (and the endpoints), we create a vector of length k with its i-coordinate being the number of intersections with e_i so far $\bmod 2$. There are $2^k + 1$ generated vectors and 2^k different vectors, so there are two vectors with the same value. In the interval between the two intervals with the same vectors, there will be an even number of intersections with every edge e_i.

Let e_j be an edge with $2h > 0$ intersection points. Then create an area close to the interval on e. Take all edge drawings exiting the right side of the area and then returning to it, use circle inversion on them and then use the axis symmetry with respect to the edge-drawing. This doesn't create any more intersection points and loses h of them. ⊞

Corollary 1. Every weakly realisable AT-graph with n edges has a representation with at most $\frac{n}{2}(2^n - 1)$ intersections.

Definition 18. Let G be a graph, $\Gamma \subseteq V \times V - V^2 : ab\Gamma cd \Leftrightarrow ((a = c \wedge bd \notin E) \vee (b = d \wedge ac \notin E)) \wedge ab \in E \wedge cd \in E$. Then by Γ^* we denote the transitive closure. We call any path on the equivalence a desire path ("pěšina"). Let $M \subseteq V \times V$. We say that M is

- sensitive, if $aMb \wedge ab\Gamma cd \Rightarrow cMd$
- transitive, if $aMb, bMc \Rightarrow aMc$
- complete, if it is sensitive and transitive
- well-behaved, if $aMb \Rightarrow ab \in E$.

By $\langle M \rangle_S$ we denote the sensitive closure, $\langle M \rangle_T$ for the transitive closure and $\langle M \rangle$ for the complete closure.

Lemma 7 (Lemma 1)**.** If $\langle M \rangle_S$ is well-behaved, then $\langle M \rangle = \langle \langle M \rangle_S \rangle_T$.

Proof. As $\langle M \rangle_S$ is well-behaved, it gives an orientation only to edges. It suffices to show that $N = \langle \langle M \rangle_S \rangle_T$ is sensitive. Assume it is not: there exists an edge $uv \in N, uw \notin N, uw \in E, vw \notin E$. This must have been created by the transitive closure, hence there exists a sequence of directed edges from u to v and at some point, there is a fork with one of the edges in $\langle M \rangle_S$ and by sensitivity, uw must be in $\langle M \rangle_S$ as well. ⊞

Lemma 8 (Lemma 2)**.** If $\langle xy \rangle_S$ is anti-symmetric, then $\langle xy \rangle_S$ is transitive (and therefore complete).

Proof. By contradiction: let $ab, bc \in \langle xy \rangle_S, ac \notin \langle xy \rangle_S$ such that the desire path from ab to bc is the shortest possible. Then we take the last uv such that $v \neq c$. Then $v \neq b$, as bc is an edge and this would violate the definition of Γ. Also, $u \neq a$, as then $ac \in \langle xy \rangle_S$.

Claim: $\forall u_i$ between u and c, there exists an edge au_i. If not, then there would be $ac \in \langle xy \rangle_S$, as $u_i c \in \langle xy \rangle_S$. This also implies $au_i \in \langle xy \rangle_S$ and $av \notin \langle xy \rangle_S$. This yields au, uv an intransitive triple with a shorter desire path. ⊞

Lemma 9 (Lemma 3)**.** If M is complete and well-behaved, $\langle xy \rangle$ is well-behaved, $xy \notin M, yx \notin M$, then $\langle M \cup \{xy\} \rangle$ is well-behaved.

Proof. If $M \cup \langle xy \rangle_S$ is sensitive and well-behaved, then from Lemma 1: $\langle M \cup \langle xy \rangle_S \rangle = \langle M \cup \langle xy \rangle_S \rangle_T$. By contradiction: let $U := \langle xy \rangle_S$ and $M \cup U$ is not well-behaved. As it is not well-behaved, there exists an oriented cycle alternating $M, U, M, U, \ldots$. We take the shortest such cycle, if there are more, we take the cycle with the shortest desire path from start to finish (the last edge that isn't incident with the first edge). All of the cycle's diagonals must be edges, but none are in $M \cup U$, else we would get a shorter cycle.

Take the first yz such that $z \neq b$ on the desire path from ab to cd (start to finish). We make five observations:

1) $a_i c \in E$, where a_i are between a and z on the desire path - if not, then $ba_i \in M$ and $ba \in M$, hence $U \cup M \neq \emptyset$ - a contradiction.

2) $zc \in E$ - otherwise $yc \in U \Rightarrow \forall i : a_i c \in U \Rightarrow$ a contradiction with $ac \notin U \cup M$

3) $zc \in M$

4) $\forall i : a_i z \in E$: if not, $a_i c \in M$, and therefore $ac \in M$ − a contradiction.

5) $az \in U$

We had a cycle *abcd*... and now we have a cycle *azcd*... with the same length, but a shorter desire path. ⊞

Theorem 17 (Characterisation of transitive orientability)**.** G is transitively orientable if and only if for every edge xy, $\langle xy \rangle_S$ is antisymmetric.

Proof. "⇒:" Obviously, there will be no contradiction there.

"⇐:" By Lemma 2, $\langle xy \rangle_S$ are complete for all xy and they are also well-behaved. Then by Lemma 3, we may connect them somehow to cover the whole graph. ⊞

List of topics